# Everything you need to know about Sequences and Series

If you knew none of the MAT137 course material before going into this class, then this is probably the easiest thing in all of MAT137. You’ve done a lot of this in high school.

In this study sheet, we’ll cover what is a sequence and how you could use it. I’ll attach many known properties of other things to sequences. We’ll see what happens if we sum every term in a sequence. This will be called a series. From there, we’ll try to find other properties of them, as well as to determine if the series will converge or diverge[[1]](#footnote-1)

## Sequences

A sequence could be thought of an ordered collection of real numbers. The concepts are very easy, so I’ll jump into definitions and name many properties of them

### Properties and definitions

***Definition***: a sequence (in ) isa a function from the natural to the reals ). For example:

We could represent every term this sequence, and more generally any sequence, as follows:

The numerical value attaches to the “a” is called the index

***Definition***: if , the natural number argument n of is called the index of .

A sequence must have infinitely many elements, though they’re not necessary unique

Additionally, they can start wherever you like, but their number of elements must go to infinity. To denote the whole sequence, the formal notation is

And informally, we write

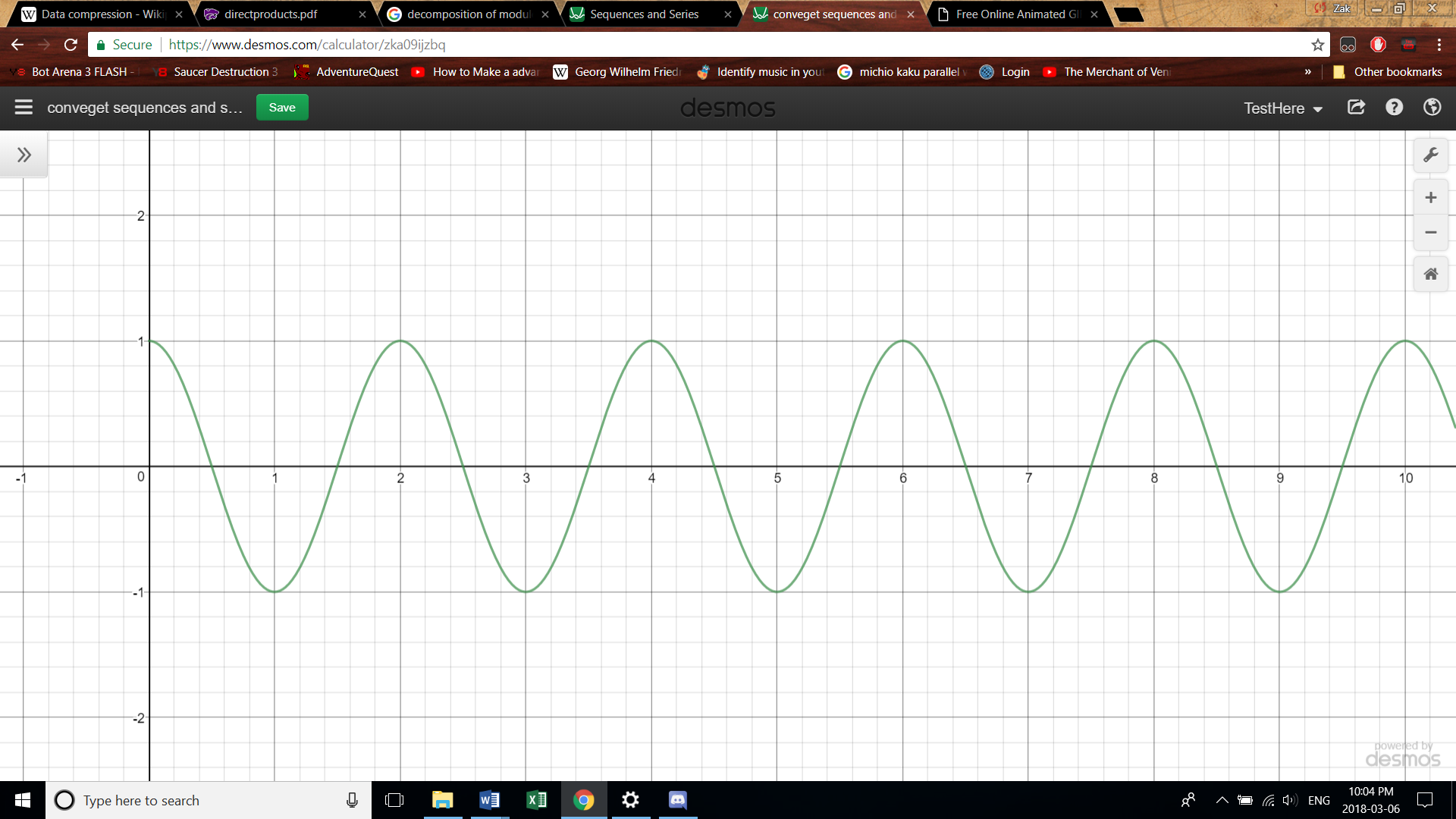
Note the different between and , as the former represents an element and the latter represents the entire sequence. Also remember that is not a set, since it’s ordered and can repeat.

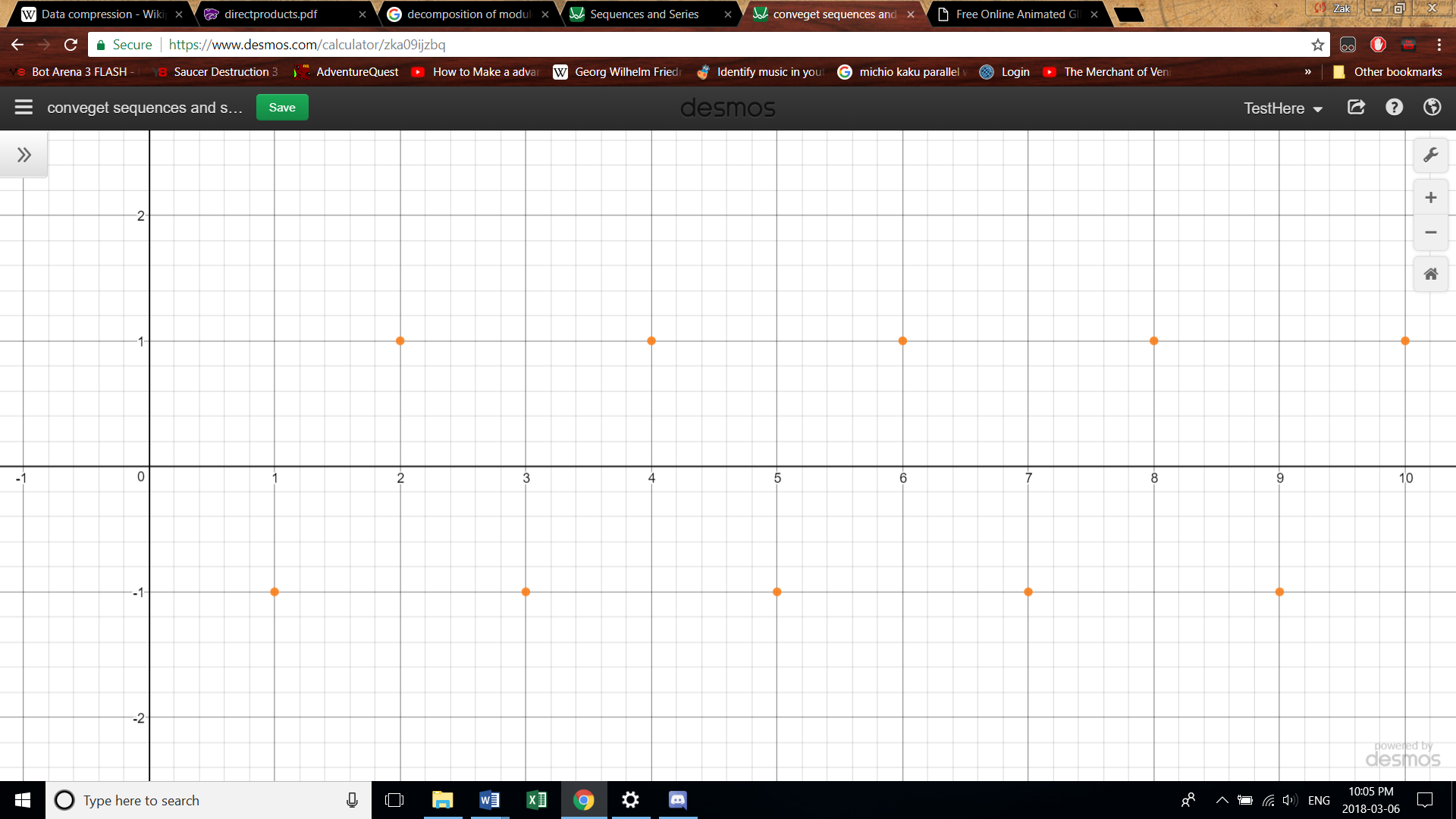
For a visual, you could play around here:

<https://www.desmos.com/calculator/za724orjzs>

### relation to functions of

if f is any function ( then it defines a unique sequence





Note that sequence could define functions, but not uniquely. Taken the previous example, the sequence could also define, were, but also any other crazy function as long as it passes those points.

This could be useful when dealing with a problem such as:

Show that the sequence given by

This would be possible with a simple comparison, but notice that this is the sequence that arises as the restriction of the function

Which could be expanded, and the derivative could be taken. The derivative is always positive for, so the sequence is increasing.

### Aspects of sequences

Definition: let be a sequence

1. We say that is strictly increasing if

And non decreasing if

1. We say that is bounded from above if such that and bounded from bellow if , and bounded if bounded from above and below

Note: a sequence cannot be bounded at a finite index like a function, since we’re always looking for all n.

There are some easy examples if we start from a function then go to a series

Proposition: if is increasing/decreasing/bounded, then is

increasing/decreasing/bounded.

Proof: I’ll prove one and leave the rest up to future you. Let the function be increasing. Therefore

Set , hence Q.E.D.

Note that the converse of this statement () is not neccesarily true, since a series doesn’t define a unique function.

### Convergent sequences

Note how we distinguish convergence and boundedness. We’ll link the two in the Monotone convergence theorem.

***Definition***: we say a sequence converges if

***Example***. If

***Proof***: Since the function as x approaches infinity converges, then

Let . DO THE REST HERE, I’M TIRED TODAY

Notationally, we write

There are exercises in the book which I recommend to do.

### Limit laws for sequences

if



Proof of 1: suppose. Let be given. Choosing s.t.

Let , if

The rest are easy to prove, and I HIGHLY recommend you do it when you’re less tired.

### Squeeze theorem for sequences

Definition: if s.t. , and

Ex. Show that W

Solution: note that

By squeeze theorem,

### Monotone convergence theorem (MCT)

This theorem relates the idea of boundedness and convergence, and is useful when you know a sequence is bounded and you want to say that it converge. This theorem links the two together.

***Theorem***: If is a non-decreasing sequence bounded from above, it converges.

***Proof***: let , which exists since is bounded from above. We want to show that

Which is equivalent to showing

We know that

Let be given and let s.t. . Since is increasing and non-decreasing, and M is an upper bound, then

***Example***. Show that Converges

***Solution***:

looking at = so if .

Or for . So is strictly decreasing. Moreover, for all n showing it’s bounded below. By MCT, the sequence converges. Q.E.D.

### Subsequences

Definition: Given a sequence , a subsequence is an increasing function[[2]](#footnote-2). This gives a new sequence where the elements come from and preserve their order.

Ex. Let . For the subsequence, take:

Which you could think of as k now being the one that increment through every natural number and we propagate the value up to “a”.

Visual:

### Relation to continuous functions

***Theorem***: Suppose , then is continuious at some point if and only if whenever is convergent, e.t.

then:

Equivalently, is continuous if

The point of this theorem

Ex :

***Proof***: proof . Suppose is continuous at “a” for all . We know that

What we want is:

We know that:

Therefore, we could choose

By our knowledge that is continuous, we could choose

Thus if

Part 2:

For this one, we’re going to prove the contrapositive. Suppose f is not continuous at “a”. We will show there’s a convergent sequence but

Since we’re assuming f is not continuous, then:

Pick which does this: set for each , and choose such that

Certainly, , but

## Series

A series is when you add every term of a sequence, or partially add terms from a partial sum.

### Definition and properties

Definition: if is a sequence, we define the series associated to to be

It’s also useful to think of part of this summand.

Definition: if is a sequence. Define a partial sequence of as

We say that the infinite series converges if the sequenceonverges. In this case.

If dones not converge, we say diverges

A couple of things to note:

1. ‘Sequences’ and ‘series’ are closely related but distinct objects. They are often confused and interchanged, so take a moment to clearly establish that a sequence is a collection of element , while a series is the sum of that sequence
2. A series is defined as a limit of finite sums, but is not a finite sum itself. You must be careful not to accidentally assume that properties of sums carry over to properties of series. An example is that finite sums are commutative, but infinite series are not (think of mathloggers video
3. Even if you know have a series converges, it can be difficult to determine the precise value to which the series converges. For example, one can show that

Despite there being no obvious trigonometric functions involved, the series converges to a number with.

***Ex***. Compute the partial sums of

***Solution***: this sum is considered the trivial partial sum:

Therefore, We say that , which could be proven using induction. For finding the sum of every elements:

Note that finding a pattern like we’ve done for the previous series is practically impossible. Here’s what Tyler had to say:

1. Computing an explicit formula for is nearly impossible

Ex.

1. It’s hard to compute the actual value. It’s easier to find that there exists a value.

Ex.

1. is a limit, **not** a sum. Sums are commutative, this is not. Think of math logger video of .

### Linearity

Theorem: if , and are convergent, and :

1. =

You could prove this using partial sums, which are commutative.

### Convergence

Theorem: if converges, then . Note that this doesn’t mean if then , as the example after the proof will show.

Proof: let and note that by the definition of a series

That previous property is extremely important for the next part. Suppose

As stated in the theorem, if the limit of the sequences converges, that doesn’t mean the series of the sequence will converge. The fundamental example of this even has a special name: The Harmonic Series. This series is so important; I’ll dedicate a special section for it

Also, if you could see the value to which it converges, you could directly proof that it converges using the definition.

### Harmonic Series

Theorem: the harmonic series doesn’t converge, i.e.

The intuitive proof of this you remember from khan academy

Formal proof: we’re going to show that

So

So showing that the harmonic series diverges

### Special Series

There aren’t many series to memorise, but these are important in many proofs, and some are simply calculation shortcuts. If you see one of these, you could show that it satisfies the properties and pull a conclusion from there.

**Geometric series**: if then

Proof: let and let:

Notice this is the factorization of . If you didn’t then distribute the left hand side, translate that to the partial sum, and eliminate the value till you get:

So

Ex. Find the value of

Solution:

**Telescoping Series:**

Ex.

So long as we’ll have

Ex.

Solution: you should know partial sums, so

So

Ex.

Solution: doing partial fraction we get

This is not quite in the form of the familiar telescoping series, but the same phenomenon applies. Develop a couple of the terms yourself to check. I have run out of time to type this so I’ll let you complete the rest (or check the book p.720).

### Comparison tests

This is just like for improper integrals. There are three main ones, the transitive, the p-test, and the limit comparison test

#### Basic comparison test (BCT)

Theorem: if for all then

1. If converges, then converges
2. The contrapositive of the first

Proof: look at improper integrals

#### P-test

The series

Converges if and only if .

You don’t need to know the proof, but it’s easy to show that since diverges, diverges by BCT. Showing it converges for values greater than 2 is also easy, but in-between is hard.

#### Limit comparison test (LCT)

If and

Then

This is also not proven, but you could look at improper integrals for more information on these in EYNTKA application of integrals. The limit comparison test is useful when the there is more than one term in the nominator or denominator, or functions with different types of growth (ex.)

Example: show that this series converges:

Solution: it is clear that this converges since the bottom’s exponent is bigger, you just need to represent this mathematically: Let be the summand and. Then:

We know that converges through the p-test, so by the limit comparison test,

Converges.

### Absolute convergence

#### Alternating series

These are series whose terms alternate from positive to negative.

Definition. Let be a sequence of non-negative real numbers. An alternating series is a series of the form

Since the signs alternate, it is likely these series converge. We only need that the series is decreasing and that approaches zero.

***Theorem (Alternating series test)***: if is a positive non-increasing sequence, the alternating series

Converges if and only if . Moreover, if the series converges with limit and is the nth partial sum, then

***Proof (Tyler p.735)***

***Example***:

***Solution***: since is non-increasing and converges to zero, by the alternating series test, the alternating series converges.

#### Absolute convergence

***Definition***: if is a sequence, the series is said to *converge absolutely* if converges.

If a series converges, but it’s absolute value doesn’t, we say it *converges conditionally*.

So, what’s the point of absolute convergence? You know that rearranging a series can produce different results (Mathloggers video on ). However, if a series is absolutely convergent, then rearranging its values produce the same result.

### Non-comparison tests

#### Integral test

The integral test converts the series into an integral, and then see if it converges or diverges.

Theorem: If f is a continuous, non-negative, non=increasing function on then

This is useful if you *see* the integral in the series.

***Example***. Show whether the series converges or diverges:

***Solution***: convert to an integral, do u substitution, and solve.

#### Ratio test

This is not for the test, so I will be brief

Let be a sequence such that

As n goes to infinity, then

1. if , then the series converges absolutely, and hence converges;
2. if , including , then the series diverges
3. if L = 1, then the test is inconclusive

This test is useful if there’s a lot of cancelling out, like:

#### Root test

This is not on the next test, so I will be brief

Let be a sequence such that

As “k” goes to infinity. Then:

1. if , the infinites series converges absolutely
2. if , the infinites series diverges
3. if , the test is inconclusive

This is useful for functions that like this:

That could be simplified by with the root test.

### Tricks

A few tricks I’ve picked up while solving convergent value:

#### Useful inequalities

Now these:

Log vs. x^n

Proof: here

.

Proof: induction

There are quite a few were an induction proof will help you. If a quotient had in its nominator and denominator a different type of growth, like.

Then you replace the nominator or denominator by whatever is simpler with an inequality that you could prove by induction, your life will become much easier. As an exercise, prove by induction that:

Is true for k sufficiently big.

Another example I’ll put is the following:

For this one, we could use the fact that

And take. This is because log(x) grows really slowly, so choose a function that mimics that.

Simplify the R.H.S., by the p-test; it converges, so the whole thing converges.

Note that you could do crazy but useless replacements, like:

But that wouldn’t help you, so make sure you’re careful with your replacements.

#### Elimination

If you get something like:

You could get rid of the like so

This could eliminate certain values to create a nicer function.

You could also eliminate function by knowing the bound of a function or by knowing that a function is increasing or decreasing, then choosing a vale. If a function is bounded (like), I’ll leave it up to future you to figure it out, since it’s really easy. For the second I’ll give an example. Figure out if:

Converges or diverges.

We know that when k is greater than or in natural number terms . This means that we could do this

We know that diverges by the p-test, so by the basic comparison test, the original summand diverges.

1. Not neccesarily finding a solution, since that is hard. [↑](#footnote-ref-1)
2. A concept not needed for MAT137 is the idea of a compact sequence. I’ll put it here for personal value: Definition: a sequence is compact if all bounded sequences has a convergent subsequence. This is related to your intuitive notion in topology, namely that a set is compact if it’s closed and bounded (and is related via the Bolzano-Weierstrass theorem) [↑](#footnote-ref-2)